Dealing with Recurrence Relations

Neil Shah

primeri.org

This handout deals with recurrence relations, and is preparation for applying these techniques on competitions at a similar level to the AMC10/12 and the AIME. Thank you to Amol Rama for proofreading and David Altizio for helping with problem selection.

§1 Recurrence Relations

Well, now that you've decided that you want to learn more about recursion and recurrence relations, it's probably a good idea to start with defining exactly what a recurrence relation is:

Recurrence Relation
A recurrence relation is a sequence where each term is defined based off of the terms before it.

Fact 1.2. Typically, a recurrence relation is defined with some sequence of values a_n where a_n denotes the nth term of the sequence. For many recurrence relations, the actual "relation" between terms is given in the form $a_n =$ something in terms of previous terms.

To make this more clear, let's look at an example:

Example 1.3
List out the first 8 terms of the recurrence relation of the sequence a_n where $a_n = a_{n-1} + 2$ for $n \geq 2$ and $a_1 = 2$.

Note that $a_1 = 2$ and each term after it is 2 more than the last so this means that the first 8 terms of the recurrence relation are just the first 8 even numbers. So, they are:

$$\boxed{2, 4, 6, 8, 10, 12, 14, 16}$$

Exercise 1.4. Find the first 8 terms of the recurrence relation for a_n when: $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$ and $a_1 = 1, a_2 = 2$.

However, it's impossible to have a complete discussion of recurrence relations without mentioning what is likely the most well-known of them all: the Fibonacci sequence.

Fibonacci

The Fibonacci sequence is the sequence of numbers a_n where:

$$a_n = a_{n-1} + a_{n-2}$$

for $n \geq 2$ with $a_0 = 1$ and $a_1 = 1$.

Essentially, you start with the numbers 1 and 1 and the next term in the sequence is always the sum of the two numbers before it.

Example 1.6

Let a_n denote the Fibonacci sequence. Find a_2, a_3, a_4, a_5.

Note that the sequence goes (starting with a_0):

$$1, 1, 1+1=2, 1+2=3, 2+3=5, 3+5=8$$

So, the desired values are:

$$\boxed{a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 8}$$

We'll come back to the Fibonacci sequence in just a little bit.

§2 Characteristic Polynomials

However, note that listing out values and doing calculations doesn't seem so efficient when a problem asks you to find the 100th or even the 500th value of a recursively defined sequence. This is precisely why we use characteristic polynomials to find the closed form of a recurrence relation. For those of you who are not familiar with the concept of a closed form, the closed form of a recurrence relation is an equation where we can define a_n in terms of n rather than in terms of the previous terms.

Characteristic Polynomial

Let a_n be a sequence where the recurrence relation is:

$$a_n = c_{n-1}a_{n-1} + \ldots + c_1 a_1 + c_0 a_0$$

for constants $c_0, c_1, \ldots c_{n-1}$. The characteristic polynomial of a_n is:

$$x^n - c_{n-1}x^{n-1} + \ldots - c_1 a_1 - c_0 a_0$$

Let $s_1, s_2, \ldots, s_n$ be the solutions when the characteristic polynomial is set equal to 0.

Theorem 2.2 (Closed Form of Recurrence Relation)

The closed form for a_k is:

$$a_k = c_1(s_1)^k + \ldots c_n(s_n)^k$$

where s_1 through s_n denote the n solutions to the characteristic polynomial.

A good example of using the characteristic polynomial of a recurrence relation to find a closed form is with the Fibonacci sequence.

Example 2.3 (Fibonacci Closed Form)
Find the closed form of the Fibonacci sequence.

Note that the recurrence relation $a_n = a_{n-1} + a_{n-2}$ maps to the characteristic polynomial

$$C(x) = x^n - x^{n-1} - x^{n-2}$$

Note that we can factor out the x^{n-2} and disregard it because the solutions that are equal to 0 just disappear from the closed form anyways. In general, each repeated root should only appear once as a^x with a as the root because instead of repeating the root, you can just modify the constant term it is multiplied by. So, let $C(x) * \frac{1}{x^{n-2}} = G(x)$. We have:

$$G(x) = x^2 - x - 1$$

We can use the quadratic formula on $G(X)$ to find that the roots are:

$$x = \frac{1 \pm \sqrt{1^2 - 4(-1)(1)}}{2 \cdot 1} = \frac{1 \pm \sqrt{5}}{2}$$

From **Theorem 2.2**, we know that:

$$a_k = (c_1)\left(\frac{1+\sqrt{5}}{2}\right)^k + (c_2)\left(\frac{1-\sqrt{5}}{2}\right)^k$$

Now, since we know that $a_2 = 2$ and $a_3 = 3$ from the answer of **Example 1.6**, we can just plug in these values to get a system of equations with 2 equations, 2 unknowns, allowing us to solve for c_1 and c_2 to complete our closed form. We have:

$$2 = (c_1)\left(\frac{1+\sqrt{5}}{2}\right)^2 + (c_2)\left(\frac{1-\sqrt{5}}{2}\right)^2$$

$$3 = (c_1)\left(\frac{1+\sqrt{5}}{2}\right)^3 + (c_2)\left(\frac{1-\sqrt{5}}{2}\right)^3$$

Solving the two above equations yields $c_1 = \frac{1}{\sqrt{5}}$ and $c_2 = -\frac{1}{\sqrt{5}}$. We will leave confirming that as an exercise to the reader. So, we have

$$a_n = \left(\frac{1}{\sqrt{5}}\right)\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

as the closed form for the Fibonacci sequence. The above is referred to as "Binet's Formula."

Here are some examples for you to gain some more familiarity with using the characteristic polynomial to find the closed forms of recurrence relations. **In general, the process for finding the closed form using the characteristic polynomial tends to be very similar to the proof for Binet's Formula above.**

Exercise 2.4. Find the closed form for the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$ where $a_0 = 3$ and $a_1 = 5$.

Exercise 2.5. Find the closed form for the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$ for $n \geq 2$ where $a_0 = 2$ and $a_1 = 7$.

§3 Generating Functions

The other way to find the closed form of a recurrence relation is using generating functions. Although this is a powerful method at finding closed forms, note that it is often easier to find closed forms using the characteristic polynomial.

Generating Function of a Recurrence Relation

The generating function for a sequence a_n based off of a recurrence relation is:

$$G(x) = a_0 + a_1 x + \ldots = \sum_{k=0}^{\infty} a_k x^k$$

Just by looking at the above definition alone, it's pretty hard to imagine how you might use this to find a closed form. Let's go over an example to maybe make that a bit more clear:

Example 3.2

Find the closed form of the recurrence relation where $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$ and $a_0 = 2$ and $a_1 = 4$.

Note that rearranging the recurrence relation yields $a_n - 4a_{n-1} + 4a_{n-2} = 0$ which means that:

$$\sum_{k=2}^{\infty} (a_n - 4a_{n-1} + 4a_{n-2}) x^n = 0$$

Note that if $f(x)$ is the generating function for a_n

$$\sum_{k=2}^{\infty} a_n x^n = f(x) - 2 - 4x$$

$$4x \sum_{k=2}^{\infty} a_{n-1} x^{n-1} = 4x(f(x) - 1)$$

$$4x^2 \sum_{k=2}^{\infty} a_{n-2} x^{n-2} = 4x^2(f(x))$$

Adding them up yields:

$$f(x) - 2 - 4x - 4xf(x) + 4x + 4x^2 f(x) = 0$$

$$f(x)(1 - 4x + 4x^2) = 2$$

$$f(x) = \frac{2}{(1-2x)^2}$$

So:

$$f(x) = 2\sum_{k=0}^{\infty} (n+1)2^n x^n$$

Thus, $a_n = 2^{n+1}(n+1)$

This is an example of how you would use the generating function to find the closed form of a recurrence relation. However, I'd just like to emphasize that if you **can** use the characteristic polynomial, use it because it's typically more efficient than this.

§4 Examples

Before we dive into the examples, I'd just like to give a few tips on solving problems with recursion and recurrence relations:

1. The first thing you should do for any problem with recurrence relations is list out the first few terms and look for a pattern. There often is one.

2. Always try simple things before getting complicated. The simple solution often exists.

Now, let's get into it. For each of the problems, I'll provide a series of steps that you can go through to try and figure it out for yourself. The examples chosen are quite instructive of techniques that often can be used to solve problems on your own.

Example 4.1 (AIME 2018 II)
Let $a_0 = 2$, $a_1 = 5$, and $a_2 = 8$, and for $n > 2$ define a_n recursively to be the remainder when $4(a_{n-1} + a_{n-2} + a_{n-3})$ is divided by 11. Find $a_{2018} \cdot a_{2020} \cdot a_{2022}$.

Try not to overcomplicate anything here.

1. List out a few values.

2. Try to finish it off from there.

Another important idea to note when solving these types of problems is to create recurrence relations of your own. Try to relate smaller versions of the problem with larger versions of this problem.

Example 4.2 (2015 AIME II)
There are $2^{10} = 1024$ possible 10-letter strings in which each letter is either an A or a B. Find the number of such strings that do not have more than 3 adjacent letters that are identical.

This one requires a bit more creativity than the previous example.

1. If you start with an n-letter string, try to relate that to the corresponding $n+1$ letter string. How do you modify the former to get the latter?

2. Generalize this pattern and come up with a recurrence relation for the situation.

3. Once you find the recurrence relation, just compute things until you find the answer.

Now, let's use an example where you now have to combine the creative ideas from examples of previous kind with the idea of using the characteristic polynomial to find the closed form. Don't be intimidated by the fact that it's a USAJMO problem. You should be able to solve it using the concepts from this handout!

Example 4.3 (2018 USAJMO)
For each positive integer n, find the number of n-digit positive integers that satisfy both of the following conditions:

- no two consecutive digits are equal, and

- the last digit is a prime.

Let's do this!

1. Split the problem into two cases. It might not be easy to see what these two cases are, but you want to make cases such that you can easily come up with a recurrence relation.
2. Find said recurrence relation using the type of creativity from the previous example.
3. Figure out a few small values for the integers that have only 1 or 2 digits.
4. Find the characteristic polynomial for the recurrence relation from step 2 and use this to find the closed form.
5. Use the small values you calculated in step 3 to find your answer.

§5 Exercises

Now that we've gone over a few examples on how to apply the idea of recurrence relations to math competition problems, here is a problem set for you to practice. Note that the problem start off quite easy, move to being doable and some of the later problems are quite hard so there should be a little bit at every difficult point.

Exercise 5.1 (2019 AMC 10A). A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

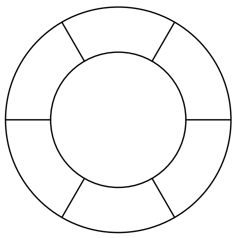
for all $n \geq 3$ Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive inegers. What is $p + q$?

Exercise 5.2 (NIMO 29). Let $\{a_n\}$ be a sequence of integers such that $a_1 = 2016$ and

$$\frac{a_{n-1} + a_n}{2} = n^2 - n + 1$$

for all $n \geq 1$. Compute a_{100}.

Exercise 5.3 (2016 AIME II). The figure below shows a ring made of six small sections which you are to paint on a wall. You have four paint colors available and will paint each of the six sections a solid color. Find the number of ways you can choose to paint each of the six sections if no two adjacent section can be painted with the same color.



Exercise 5.4 (2020 AMC12A). Let (a_n) and (b_n) be the sequences of real numbers such that

$$(2+i)^n = a_n + b_n i$$

for all integers $n \geq 0$, where $i = \sqrt{-1}$. What is

$$\sum_{n=0}^{\infty} \frac{a_n b_n}{7^n}\ ?$$

Exercise 5.5 (HMMT 2007 Algebra). An infinite sequence of positive real numbers is defined by $a_0 = 1$ and $a_{n+2} = 6a_n - a_{n+1}$ for $n = 0, 1, 2, \cdots$. Find the possible value(s) of a_{2007}.

Exercise 5.6 (2001 AIME I). A mail carrier delivers mail to the nineteen houses on the east side of Elm Street. The carrier notices that no two adjacent houses ever get mail on the same day, but that there are never more than two houses in a row that get no mail on the same day. How many different patterns of mail delivery are possible?

Exercise 5.7. Compute the number of rearrangements $a_1, a_2, \ldots, a_{2018}$ of the sequence $1, 2, \ldots, 2018$ such that $a_k > k$ for *exactly* one value of k.

Exercise 5.8 (HMMT 2016 Team). A nonempty set S is called *well-filled* if for every $m \in S$, there are fewer than $\frac{1}{2}m$ elements of S which are less than m. Determine the number of well-filled subsets of $\{1, 2, \ldots, 42\}$.