Applications of Symmetric Sums

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§1 Introduction

Vieta's Formulas are an *amazing* trick that allows us to learn things about the roots of polynomials. Problem writers on the AMC and AIME frequently write problems that can be solved using these techniques and imagine how it feels to be able to solve problems without actually having to find the solutions to the equations: It almost feels like cheating!

§1.1 Some Notation

Symmetric Notation

A sum of terms which is symmetric over all included variables can be written in the form:

$$\sum_{\text{sym}} \text{term}$$

where the word "term" is replaced with the actual term that the variables are symmetric over.

"Symmetric" Polynomials

A polynomial is "symmetric" when you can permute the values of the variables and the polynomial would remain the same. For example, in $x + y + z$ you could permute the variables and still retain the same expression.

Here's an example to make this more clear:

Example 1.1

Expand the expression when we are working with the three variables x, y, z:

$$\sum_{\text{sym}} x^2 y$$

Solution. Based on the above definition of how this notation works, we know that this will be the sum of all possible terms that contain one variable squared along with another. So, the sum expands to:

$$x^2y + x^2z + xy^2 + y^2z + xz^2 + yz^2$$

□

Remark 1.2. I recommend using this notation to refer to symmetric polynomials and sums whenever you are dealing with them. It reduces the clutter in your work by decreasing the number of terms. This in turn can often make it easier to make key observations about a problem because there are less distractions.

§1.2 Elementary Symmetric Sums

Although many different examples of symmetric polynomials exist, with many of them being so complicated, it's always true that symmetric polynomials can be expressed in terms of "simple" polynomials. If you need some convincing, here's an exercise in terms of two variables:

Example 1.3
Express the symmetric polynomials below in terms of $x+y$ and xy:

1. $xy^2 + x^2y$
2. $x^2 + y^2$

Solution. Here we go:

1. The first one is quite simple. All we have to do is factor out the common xy. We get:
$$xy(x+y)$$
2. The second one is a bit more complicated. However, we know that if we want to get the sum of individual variables, the only way to get this is by squaring $x+y$. Squaring xy or multiplying $x+y$ with xy yields terms with multiple variables. However, $(x+y)^2 = x^2 + 2xy + y^2$. We can subtract $2xy$ from this in order to get our desired expression. So:
$$x^2 + y^2 = (x+y)^2 - 2xy$$

□

We can generalize the above to multiple variables as well. This idea is known as the "elementary symmetric sums."

Elementary Symmetric Sums
Assume that we are dealing with the variables $x_1, x_2, \ldots, x_k$. For positive integer n such that $n \le k$, let s_n denote the n^{th} elementary symmetric sum of the set of k variables. Then, we have:

$$s_n = \sum_{1 \le a_1 \le \ldots \le a_n \le k} x_{a_1} \ldots x_{a_n} = \sum_{\text{sym}} x_{a_1} \ldots x_{a_n}$$

Remark 1.4. We have two expressions for the n^{th} symmetric sum where one is written out formally and the other is written using the notation from the first section.

This is a pretty formal definition of elementary symmetric sums. However, there's a more intuitive way to think about this: the n^{th} elementary symmetric sum is the sum of the products of every possible combination of n variables out of the k variables we are dealing with.

Example 1.5
What are the elementary symmetric sums of the three variables x, y, z?

Solution. From our definition, s_1 is the sum of all combinations of 1 distinct variable. So, it must be true that $s_1 = x + y + z$. Similarly, $s_2 = xy + xz + yz$ and $s_3 = xyz$. □

Remark 1.6. In general, when we are dealing with k variables, s_1 is the sum of all k variables and s_k is the product of all k variables.

We're going to go over one more example to ensure that the concept of elementary symmetric sums is driven home:

Example 1.7
What are the elementary symmetric sums of the four variables a, b, c, d?

Note that we have:

$$s_1 = a + b + c + d$$
$$s_2 = ab + ac + ad + bc + bd + cd$$
$$s_3 = abc + abd + acd + bcd$$
$$s_4 = abcd$$

Fact 1.8. All symmetric polynomials (polynomials that are symmetric relative to each variable in them) are expressible in terms of the elementary symmetric sums.

Remark 1.9. When solving problems, if you have symmetric polynomials, it is often a good idea to express them in terms of the elementary symmetric sums if you know the values of any of them.

§2 Vieta's Formulas

Elementary symmetric sums and Vieta's Formulas are two concepts that go hand-in-hand. It's nearly impossible to think about one without thinking of the other. Essentially, Vieta's Formulas are a tool that allows us to relate the roots of polynomials to elementary symmetric sums.

§2.1 A Small Case

Let's start by taking a look at what Vieta's Formulas look like when applied to quadratics:

Theorem 2.1 (Vieta's Formulas on Quadratics)
Given any quadratic equation written in the form

$$P(x) = ax^2 + bx + c$$

it is always true that $s_1 = -\frac{b}{a}$ and $s_2 = \frac{c}{a}$, where s_n denotes the n^{th} symmetric sum of the roots r and s of the polynomial P.

This might look complicated to those of you who have never seen it before, but it's actually just a formulation of a very simple idea.

Proof. We're trying to prove a result about the roots of a quadratic. So, let's start by expressing the quadratic in terms of its roots. Given r and s denote the two not necessarily distinct roots of $P(x)$, we have:

$$P(x) = ax^2 + bx + c = a(x - r)(x)$$

If we expand this out, we have:

$$P(x) = a(x - r)(x - s) = ax^2 - a(r + s)x + ars$$

Equating the coefficients of both representations of P gives $b = -a(r + s)$ and $ars = c$. It follows from there that $s_1 = -\frac{b}{a}$ and $s_2 = \frac{c}{a}$. □

Let's look at an example.

Example 2.2 (AMC10A 2003)
What is the sum of the reciprocals of the roots of the equation $\frac{2003}{2004}x + 1 + \frac{1}{x} = 0$?

Solution. Since we are asked to find some information about the roots, we should immediately know that we need to use Vieta's Formulas in some capacity. A good place to start might be to put this equation in a form where we can apply Vieta's Formulas: a quadratic. Let's start by multiplying both sides of the equation by x in order to get a quadratic equation:

> *Remark 2.3.* Note that we can only do this because 0 is not a solution to our quadratic equation (quite obviously because we are dividing by x in one of the terms). If 0 was a solution, we would not be able to do this because multiplying both sides of the equation by 0 would not yield any results.

$$\frac{2003}{2004}x^2 + x + 1 = 0$$

So, by Vieta's Formulas, we know that if r and s are the two roots of the quadratic:

$$r + s = -\frac{2004}{2003}$$

$$rs = \frac{2004}{2003}$$

Now, let's see how we can express the sum of the reciprocals of the roots in terms of the elementary symmetric sums of the roots. We have:

$$\frac{1}{r} + \frac{1}{s} = \frac{r+s}{rs}$$

We know $r + s$ and rs so from here, we can just plug in our known values to get that the sum of the reciprocals of the roots is

$$\frac{1}{r} + \frac{1}{s} = \frac{r+s}{rs} = -\frac{2004}{2003} \cdot \frac{2003}{2004} = \boxed{-1}.$$

□

§2.2 Generalizing

Now that the general point is across, let's generalize this to polynomials of all positive integer degrees, rather than just quadratics:

Theorem 2.4 (Vieta's Formulas on All Polynomials)
Given any polynomial written in the form:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

for positive integer n, we have:

$$s_k = (-1)^k \frac{a_{n-k}}{a_n}$$

Similar to the proof for Vieta's Formulas on the smaller case we looked at, this proof also follows similarly (e.g. set up the equations and expand).

Proof. Just like last time, since we are looking at the relationship between the roots of a polynomial and the coefficients, let's express the polynomial in both ways:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = a_n(x - r_1)\ldots(x - r_n)$$

Note that if we want to isolate the coefficient of a particular x^k term, we want to choose the x in the expression $x - r_{\text{something}}$ in k of the n expressions and (-1) times the root in the other $n - k$ expressions.

From the above, it follows that the coefficient of the x^k term is the sum of all combinations of negative roots with size $n - k$. So, the coefficient of x^k winds up as $(-1)^{n-k} s_{n-k} a_n$. This is the statement of Vieta's Formulas, so we are done. □

> *Remark 2.5.* The proof above is similar to the proof of the Binomial Theorem.

There is one last thing about Vieta's Formulas that is often confusing for people using it for the first time. What happens if I'm looking at a polynomial and one of the terms is missing? Let's take a look at one such scenario:

Example 2.6
Find the elementary symmetric sums of the roots of the polynomial $P(x) = 2x^3 + 6$.

Solution. Since there aren't any x^2 or x terms, it might be tempting to think that the first and second elementary symmetric sums of the roots of P don't exist. However, instead rewrite the original polynomial as

$$P(x) = 2x^3 + 6 = 2x^3 + 0x^2 + 0x + 6.$$

Basically, the important thing to realize is that a term being missing is equivalent to the coefficient being 0. Using this, we have:

$$s_1 = -\frac{0}{2} = 0, s_2 = \frac{0}{2} = 0, s_3 = -\frac{6}{2} = -3.$$

□

While the idea behind Vieta's Formulas may seem super simple, and it is, this simple technique plays a large role in solving many difficult problems as we'll see later during the sections on Examples and Exercises.

§3 Newton Sums

Newton Sums is another technique that is very helpful in solving math competition problems. It allows us to express power sums like $r_1^n + r_2^n + \ldots + r_k^n$ in terms of the elementary symmetric sums. Similar to how we dealt with Vieta's Formulas, we will look at smaller degree polynomials before generalizing to polynomials in general.

§3.1 A Small Case

Example 3.1
Let a polynomial $P(x) = x^2 - 2x + 6$ with (complex) roots a and b. Then what is the value of $a^2 + b^2$?

Solution. The important realization to make here is that $a^2 + b^2 = (a + b)^2 - 2ab$. We can use Vieta's Formulas to find $a + b$ and ab individually and then we can put these together to find the desired value. By Vieta's Formulas, we can find that $a + b = \frac{2}{1} = 2$ and $ab = \frac{6}{1} = 6$. So:

$$a^2 + b^2 = (a + b)^2 - 2ab = 2^2 - 2 * 6 = 4 - 12 = \boxed{-8}$$

□

The idea we used in the above example is essentially the same as the idea for Newton Sums. Basically, you can express the power sums in terms of the elementary symmetric sums, and the relationship between the two can be expressed similarly for each power sum.

Theorem 3.2 (Newton Sums on Quadratics)
Let s_k denote the kth elementary symmetric sum and let p_k denote the kth power sum (the sum of each term to the kth power). Then:

$$p_k = p_{k-1}s_1 - p_{k-2}s_2$$

for each $k \geq 3$. For p_1 and p_2, we have:

$$p_1 = s_1$$

$$p_2 = p_1 s_1 - 2s_2$$

Note that in this case we are dealing with the roots of quadratics, meaning that we only have 2 variables to deal with.

Let's put the above theorem into practice:

Example 3.3
Check the Newton Sums for p_2 and p_3.

Solution. From the above, it suffices to show that:

$$p_3 = p_2 s_1 - p_1 s_2$$

Since we are dealing with quadratics, let $s_1 = x + y$ and $s_2 = xy$. Let's start with p_2. We have:

$$p_2 = (x + y)^2 - 2xy = x^2 + 2xy + y^2 - 2xy = x^2 + y^2$$

So, the definition of p_2 is correct. Let's try p_3:

$$p_3 = (x^2 + y^2)(x + y) - (x + y)(xy) = x^3 + x^2y + xy^2 + y^3 - x^2y - xy^2 = x^3 + y^3$$

Thus, the Newton Sum for p_3 works as well. □

§3.2 One More Small Case

Theorem 3.4 (Newton Sums for Cubics)
Assume the same definitions for s_k and p_k as the previous version. For $k \leq 3$ we have:

$$p_1 = s_1, p_2 = s_1 p_1 - 2s_2, p_3 = s_1 p_2 - p_1 s_2 + 3s_3$$

For larger k, we have:

$$p_k = s_1 p_{k-1} - s_2 p_{k-2} + s_3 p_{k-3}$$

Note that in this case we are dealing with cubics, so we only have 3 variables.

Something to note is that the maximum number of distinct terms in a newton sum is the number of variables being used.

Exercise 3.5. Check that the theorem above holds for p_3 and p_4.

> *Remark 3.6.* This may be computationally heavy but the reason for p_4 is to demonstrate the cap on the number of terms in the newton sum.

Example 3.7 (AMC12A 2019)
Let p_k denote the sum of the k^{th} powers of the roots of the polynomial $x^3 - 5x^2 + 8x - 13$. In particular, $p_0 = 3$, $p_1 = 5$, and $p_2 = 9$. Let a, b, and c be real numbers such that $p_{k+1} = a\, p_k + b\, p_{k-1} + c\, p_{k-2}$ for $k = 2, 3, ...$ What is $a + b + c$?

Solution. Since we know that the coefficients in the definition of the power sum are constant for $k \geq 2$, we can just express p_3 using Newton Sums to find our answer. From our above Theorem, we have:

$$p_3 = s_1 p_2 - s_2 p_1 + 3s_3$$

Note that $p_0 = 3$ so we can substitute that in. We have:

$$p_3 = s_1 p_2 - s_2 p_1 + s_3 p_0$$

We want to find the sum of the coefficients of the power sums, so we essentially want to find $s_1 - s_2 + s_3$. By Vieta's Formulas on the given polynomial, we can find that $s_1 = 5$, $s_2 = 8$, and $s_3 = 13$. So:

$$a + b + c = s_1 - s_2 + s_3 = 5 - 8 + 13 = \boxed{10}$$

□

§3.3 Generalizing

Without further ado, let's look at the generalization of Newton Sums to polynomials of all positive integer degrees:

Theorem 3.8 (General Form of Newton Sums)
Let k be the number of variables we are dealing with and define s_k and p_k as we did previously. When $i < k$, we have:

$$p_1 = 1 \cdot s_1$$

$$p_2 = p_1 s_1 - 2s_2$$
$$p_3 = p_2 s_1 - p_1 s_2 + 3s_3$$
$$\vdots$$
$$p_{k-1} = p_{k-2}s_1 - p_{k-3}s_2 + \ldots + (-1)^{k-2} s_{k-1}(k-1)$$

For $i > k$, we have:

$$p_i = p_{i-1}s_1 - p_{i-2}s_2 + \ldots + (-1)^{k-1} p_{i-k}s_k$$

Remark 3.9. Looking at the above formulation of Newton Sums reveals that the quadratic and cubic cases are just specific sub-cases of the generalization, as expected.

This is a particularly powerful result, because this allows us to relate the power sums of multiple variables to their elementary symmetric sums, which we in turn can usually find in problems using Vieta's Formulas.

§4 Examples

The Examples section will be split into two subsections, Vieta's Formulas and Newton Sums. Note that the section on Vieta's Formulas will not require knowledge of Newton Sums but the section on Newton Sums may require some prerequisite knowledge of how Vieta's Formulas work.

§4.1 Vieta's Formulas

Let's first start off with a standard application. The process of creating and then solving equations is important in solving Vieta's problems.

Example 4.1 (AIME II 2014)
Real numbers r and s are roots of $p(x) = x^3 + ax + b$, and $r + 4$ and $s - 3$ are roots of $q(x) = x^3 + ax + b + 240$. Find the sum of all possible values of $|b|$.

Solution. In this situation, we have information about the roots of the polynomial and about the coefficients of the polynomial. This is basically SCREAMING for us to use Vieta's Formulas. Let's preemptively apply them and figure out how to use them later. Let s_{pn} denote the n^{th} elementary symmetric sum of the roots of p and s_{qn} denote the n^{th} elementary symmetric sum of the roots of q. We have:

$$s_{p1} = 0, s_{p2} = a, s_{p3} = -b$$

$$s_{q1} = 0, s_{q2} = a, s_{q3} = -b - 240$$

Note that we've only been given information about two of the roots of each of the polynomials. A good place to start appears to be finding information about the third roots. Since we know that the sum of the roots for both polynomials is equal to 0 from the above symmetric sums, the roots of p must be $r, s, -r-s$ and the roots of q must be $r+4, s-3, -r-s-1$. The other information we can glean from Vieta's Formulas is that the roots of the two polynomials have equivalent second symmetric sums. Equating the two yields:

$$rs + (r+s)(-r-s) = (r+4)(s-3) + (-r-s-1)(r+s+1)$$

Simplifying the above yields:

$$s = \frac{13+5r}{2}$$

We also know the relationship between the third symmetric sums, which is that the third symmetric sum of q is 240 more than the third symmetric sum of p. This yields:

$$rs(-r-s) + 240 = (r+4)(s-3)(-r-s-1)$$

Plugging in our expression of s in terms of r into the above equation and simplifying further yields $(r,s) = (-5,-6), (1,9)$ which correspond to $|b| = 330, 90$ and an answer of $\boxed{420}$. □

Let's go over one more example of Vieta's Formulas, this one from the recent AOIME (a.k.a. the 2020 AIME II).

Example 4.2 (AIME II 2020)
Let $P(x) = x^2 - 3x - 7$, and let $Q(x)$ and $R(x)$ be two quadratic polynomials also with the coefficient of x^2 equal to 1. David computes each of the three sums $P+Q$, $P+R$, and $Q+R$ and is surprised to find that each pair of these sums has a common root, and these three common roots are distinct. If $Q(0) = 2$, then $R(0) = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Using what we know, we can set up the three equations for P, Q, R for real a, b, c:

$$P(x) = x^2 - 3x - 7$$
$$Q(x) = x^2 + ax + 2$$
$$R(x) = x^2 + bx + c$$

So, our sums are:

$$P + Q = 2x^2 + (a-3)x - 5$$
$$P + R = 2x^2 + (b-3)x + c - 7$$
$$Q + R = 2x^2 + (a+b)x + c + 2$$

Now that we have the sums of the polynomials, let's apply what we know from the problem statement. Let the common root of $P+Q$ and $P+R$ be p, $P+Q$ and $Q+R$ be q and the common root of $P+R$ and $Q+R$ be r. So, we have:

$$P + Q = 2(x-p)(x-q)$$
$$P + R = 2(x-p)(x-r)$$
$$Q + R = 2(x-q)(x-r)$$

We have information about the coefficients of the polynomials so our natural next step is to apply Vieta's Formulas:

$$p+q=-\frac{a-3}{2}, p+r=-\frac{b-3}{2}, q+r=-\frac{a+b}{2}$$

$$pq=-\frac{5}{2}, pr=\frac{c-7}{2}, qr=\frac{c+2}{2}$$

Note that we have 6 equations along with 6 unknowns which provides us with enough information to solve for our variables. Solving out the equations yields $c=\frac{52}{19}$ which leads to an answer of $\boxed{71}$. □

§4.2 Newton Sums

The last two examples were regarding Vieta's Formulas. This next one is an application of Newton Sums that demonstrates how powerful the idea is. Don't be intimidated by the fact that this is a USAMO problem. It's a rather old problem and is actually easier than many modern-day AIME problems.

Example 4.3 (USAMO 1973)
Determine all roots, real or complex, of the system of simultaneous equations

$$\begin{aligned} x+y+z&=3,\\ x^2+y^2+z^2&=3,\\ x^3+y^3+z^3&=3. \end{aligned}$$

Solution. These three equations look oddly familiar! They're just the first three power sums. So, we have use Newton's Sums to write these out in terms of the elementary symmetric sums:

$$s_1=3$$

$$s_1^2-2s_2=3$$

$$s_1p_2-s_2p_1+3s_3=3$$

We can also make some more substitutions to completely get rid of p_2 and p_1. We can replace p_2 by using Newton's Sums once again and we can just replace all mentions of p_1 with s_1 because they mean the same thing:

$$s_1=3$$

$$s_1^2-2s_2=3$$

$$s_1^3-3s_1s_2+3s_3=3$$

Solving this yields $s_2=3$ and $s_3=1$. Since we want to find the solutions for x,y,z, it might make sense to work backwards on Vieta's Formulas to put these symmetric sums into a polynomial, as finding the roots of a polynomial is something we should know how to do in most cases. Going backwards on Vieta's Formulas yields that the polynomial f such that x,y,z are the roots is:

$$f(x)=x^3-3x^2+3x-1$$

This factors to $(x-1)^3$, so the only solution to the system of simultaneous equations is $\boxed{(x,y,z)=(1,1,1)}$. □

§5 Exercises

Exercise 5.1 (USMCA Challenger 2020). Let a, b, c, d be the roots of the quartic polynomial $f(x) = x^4 + 2x + 4$. Find the value of

$$\frac{a^2}{a^3+2} + \frac{b^2}{b^3+2} + \frac{c^2}{c^3+2} + \frac{d^2}{d^3+2}$$

Exercise 5.2 (AIME 1996). Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, and c, and that the roots of $x^3 + rx^2 + sx + t = 0$ are $a + b, b + c$, and $c + a$. Find t.

Exercise 5.3 (AIME I 2001). Find the sum of the roots, real and non-real, of the equation $x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0$, given that there are no multiple roots.

Exercise 5.4 (David's Problem Stash 2). Let a, b, and c be nonzero real numbers such that $a + b + c = 0$ and

$$28(a^4 + b^4 + c^4) = a^7 + b^7 + c^7.$$

Find $a^3 + b^3 + c^3$.

Exercise 5.5 (AIME I 2019). For distinct complex numbers $z_1, z_2, \ldots, z_{673}$, the polynomial

$$(x - z_1)^3 (x - z_2)^3 \cdots (x - z_{673})^3$$

can be expressed as $x^{2019} + 20x^{2018} + 19x^{2017} + g(x)$, where $g(x)$ is a polynomial with complex coefficients and with degree at most 2016. The value of

$$\left| \sum_{1 \le j < k \le 673} z_j z_k \right|$$

can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Exercise 5.6 (AIME II 2018). Let r, s, and t be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Find $(r + s)^3 + (s + t)^3 + (t + r)^3$.

Exercise 5.7 (AIME II 2003). Consider the polynomials $P(x) = x^6 - x^5 - x^3 - x^2 - x$ and $Q(x) = x^4 - x^3 - x^2 - 1$. Given that z_1, z_2, z_3, and z_4 are the roots of $Q(x) = 0$, find $P(z_1) + P(z_2) + P(z_3) + P(z_4)$.

Exercise 5.8 (Winter OMO 2011-2012). Let a, b, c be the roots of the cubic $x^3 + 3x^2 + 5x + 7$. Given that P is a cubic polynomial such that $P(a) = b + c$, $P(b) = c + a$, $P(c) = a + b$, and $P(a + b + c) = -16$, find $P(0)$.

Exercise 5.9 (CMIMC 2019). Let a, b and c be the distinct solutions to the equation $x^3 - 2x^2 + 3x - 4 = 0$. Find the value of

$$\frac{1}{a(b^2+c^2-a^2)} + \frac{1}{b(c^2+a^2-b^2)} + \frac{1}{c(a^2+b^2-c^2)}.$$

Exercise 5.10 (SMT 2010). Find the sum of all solutions of the equation

$$\frac{1}{x^2-1} + \frac{2}{x^2-2} + \frac{3}{x^2-3} + \frac{4}{x^2-4} = 2010x - 4$$

Exercise 5.11 (Winter OMO 2011-2012). Let S denote the sum of the 2011th powers of the roots of the polynomial $(x-2^0)(x-2^1)\cdots(x-2^{2010}) - 1$. How many ones are in the binary expansion of S?

Exercise 5.12 (AIME II 2015). Let x and y be real numbers satisfying $x^4y^5 + y^4x^5 = 810$ and $x^3y^6 + y^3x^6 = 945$. Evaluate $2x^3 + (xy)^3 + 2y^3$.

Exercise 5.13 (HMMT 2007). The complex numbers α_1, α_2, α_3, and α_4 are the four distinct roots of the equation $x^4 + 2x^3 + 2 = 0$. Determine the unordered set

$$\{\alpha_1\alpha_2 + \alpha_3\alpha_4, \alpha_1\alpha_3 + \alpha_2\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3\}.$$

Exercise 5.14 (CMIMC 2016). Let $r_1, r_2, \ldots, r_{20}$ be the roots of the polynomial $x^{20} - 7x^3 + 1$. If

$$\frac{1}{r_1^2+1} + \frac{1}{r_2^2+1} + \cdots + \frac{1}{r_{20}^2+1}$$

can be written in the form $\frac{m}{n}$ where m and n are positive coprime integers, find $m+n$.

Exercise 5.15 (CMIMC 2017). The polynomial $P(x) = x^3 - 6x - 2$ has three real roots, α, β, and γ. Depending on the assignment of the roots, there exist two different quadratics Q such that the graph of $y = Q(x)$ pass through the points (α, β), (β, γ), and (γ, α). What is the larger of the two values of $Q(1)$?